ARE 210: Oct. 27, 2021 Section Notes

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Final 2018: Q3

Setup: Suppose that $\{X_i\}_{i=1}^{\infty}$ are a sequence of iid random variables and $X_i \in (-1, 1)$ with density

$$f(x, \theta) = \frac{1}{2}(1 + \theta x)\mathbb{I}(x \in (-1, 1))$$

a) Show θ is identified. b) Write down the sample log-likelihood and the first order conditions to characterize the MLE of θ.

Solution:
Strategy: Part a) proceeds like many identification problems. What we want to show is that

$$\theta_1 \neq \theta_2 \Rightarrow \exists x^* \ s.t. \ f(x^*, \theta_1) \neq f(x^*, \theta_2)$$

That is, if $\theta_1 \neq \theta_2$, then the densities are different. Usually it's easier to show the (logically equivalent) contrapositive

$$\forall x^* \ f(x^*, \theta_1) = f(x^*, \theta_2) \Rightarrow \theta_1 = \theta_2$$

So we start by assuming that the densities are the same everywhere and prove that it implies $\theta_1 = \theta_2$.

Part b) is also typical of MLE problems. We'll write down the likelihood as the product of the density at all of the realized values, take logs, and then take the derivative with respect to θ to get the first order condition. Because it ends up having a fairly tricky form, we'll just stop there rather than rewrite $\hat{\theta}_{MLE}$ in terms of the data.

Step 1: Identification
As mentioned in the strategy, we're going to start by assuming we have identical densities for two (possibly equal) values of theta

$$\forall x^* \ f(x^*, \theta_1) = f(x^*, \theta_2) \Rightarrow \forall x \ \frac{1}{2}(1+\theta_1 x)\mathbb{I}(x \in (-1, 1)) = \frac{1}{2}(1+\theta_2 x)\mathbb{I}(x \in (-1, 1))$$

We can subtract the right-hand side from the left to get

$$\forall x\ \frac{1}{2}\mathbb{I}(x \in (-1,1))(x(\theta_1 - \theta_2)) = 0$$

Now we note that for any $x \in (-1,1)$ and $x \neq= 0$, this expression implies

$$\theta_1 = \theta_2$$

So θ is point identified.

Step 2: Write down log-likelihood

The general form for the likelihood function for an iid sample is $L(\{x_i\}_{i=1}^n, \theta) = \prod_{i=1}^{n} f(x_i, \theta)$. Note that this is the density (rather than cumulative density), it depends on the choice of θ, and we're using realized values rather than random variables. In this problem, the likelihood function is

$$L(\{x_i\}_{i=1}^n, \theta) = \prod_{i=1}^{n} \frac{1}{2}(1 + \theta x_i)\mathbb{I}(x_i \in (-1,1))$$

Generally, it's easier to work with the log-likelihood function since it turns the product into a sum of logs and to drop the indicator function since the probability of observing $x_i \notin (-1,1)$ is 0.

$$\begin{aligned}\log(L(\{x_i\}_{i=1}^n, \theta)) &= \log\left(\prod_{i=1}^{n} \frac{1}{2}(1 + \theta x_i)\mathbb{I}(x_i \in (-1,1))\right) \\ &= n\log\left(\frac{1}{2}\right) + \sum_{i=1}^{n} \log(1 + \theta x_i)\end{aligned}$$

Step 2: Find the FOC

We now take the derivative with respect to θ and set it equal to 0 to get the FOC

$$\frac{\partial \log(L(\{x_i\}_{i=1}^n, \theta))}{\partial \theta} = \sum_{i=1}^{n} \frac{x_i}{1 + \hat{\theta}_{MLE} x_i} = 0$$

Which completes the problem. To actually implement this estimator we'd have to solve for $\hat{\theta}_{MLE}$, either analytically or numerically.

Final 2018: Q1a (slightly altered)

Setup: Consider the density function

$$p(x, \theta, \nu) = \frac{\theta \nu^\theta}{x^{\theta+1}}\mathbb{I}(x \geq \nu)$$

does $p(.,.,.)$ belong to the exponential family? What if we omit the indicator?

Solution:
Strategy: The answer to the first question is "No, because the support depends on a parameter." When that is not the case, the solution strategy is to try to rewrite the density in terms of the exponential family. You will generally either succeed or find it's impossible, which will lead to the answer.

Step 1: Write out the form for the exponential family and assign parts
Recall that the exponential family is characterized by a density of the form

$$f(x|\theta,\nu) = h(x)c(\theta,\nu)\exp\left(\sum_{i=1}^{k} w_i(\theta,\nu)t_i(x)\right)$$

We can see that $c(\theta,\nu) = \theta\nu^{\theta}$. We can't, however, write $h(x) = x^{-(\theta+1)}$ since that depends on θ. Instead, we'll have to see if we can write it in the exponential part.

Note that $x^{-(\theta+1)} = \exp(\log(x^{-(\theta+1)}))$. So we can write $w_1(\theta,\nu) = -(\theta+1)$ and $t_1(x) = \log(x)$. Finally, we'll just assign $h(x) = 1$. So we have

$$p(x,\theta,\nu) = 1 * \theta\nu^{\theta}\exp(-(\theta+1)log(x))$$

so $p(.,.,.)$ belongs to the exponential family.