ARE 210: Sep. 14, 2022 Section Notes

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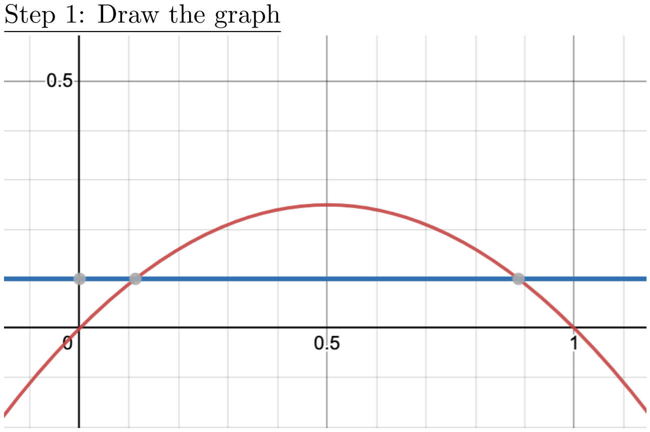
1 PS1 Q8

Setup: Suppose $X \sim U[0,1]$ and define $Y = X - X^2$

1. derive the density of Y and graph it
2. derive the CDF of Y and graph it
3. What is the median of Y?

Solution:
Strategy: I think it's actually helpful to start with the CDF of Y. It's always helpful to draw the transformation if you can because it can help you identify tricky issues with these sorts of problems. In this problem, because it's not monotonic you either have to break it up into two transformations or solve for the CDF first. I think the latter is preferable. After finding the CDF it's easy to calculate the density (derivative) and median (set it equal to 0.5).

Step 1: Draw the graph



The transformation is an upside-down parabola. For a given y (the blue line)there are two segments of the domain of X for which $Y \leq y$. The probability that $Y \leq y$ is the sum of the probabilities associated with X being in these two segments.

Step 2: Solve for the two intersection points

For a given y we're looking for $-X^2 + X - y = 0$. Using the quadratic formula, we get $X = (1 \pm \sqrt{1-4y})/2$

<u>Step 3: Solve for F_Y</u>

From the graph, we know $F_Y(y) = \mathbb{P}(X \le (1-\sqrt{1-4y})/2) + \mathbb{P}(X \ge (1+\sqrt{1-4y})/2)$ for $y \in [0, 0.25]$. Since $X \sim U[0,1]$, we know $\mathbb{P}(X \le (1-\sqrt{1-4y})/2) = (1-\sqrt{1-4y})/2$. Similarly, $\mathbb{P}(X \ge (1+\sqrt{1-4y})/2) = 1-(1+\sqrt{1-4y})/2 = (1+\sqrt{1-4y})/2$. So $F_Y(y) = (1-\sqrt{1-4y})/2 + (1+\sqrt{1-4y})/2 = 1-\sqrt{1-4y}$.

<u>Step 4: Take derivative to get f_Y</u>

Now that we know $F_Y(y) = 1-\sqrt{1-4y}$, we just need to take its derivative to get the density. So $f_Y(y) = 2(1-4y)^{-1/2}$.

<u>Step 5: Solve for median</u>

Given the CDF, we're looking for y^* such that $F_Y(y^*) = 0.5$. This is given by

$$\begin{aligned}
F_Y(y^*) &= 0.5 \\
&\Rightarrow 1-\sqrt{1-4y^*} = 0.5 \\
&\Rightarrow \sqrt{1-4y^*} = 0.5 \\
&\Rightarrow y^* = 3/16
\end{aligned}$$

2 Integral Example 1

Setup: We have a collection of sets $\Omega = \{A, B, C, D\}$, which is associated with a σ algebra 2^Ω and a probability distribution $\mathbb{P}$ given by

- $\mathbb{P}(A) = 0.25$
- $\mathbb{P}(B) = 0.1$
- $\mathbb{P}(C) = 0.6$
- $\mathbb{P}(D) = 0.05$

We also have a function $g : \Omega \to \mathbb{R}$ given by

- $g(A) = 8$
- $g(B) = 25$
- $g(C) = 5$
- $g(D) = 5$

What is $\mathbb{E}[g(\omega)]$? What is $\int_{\{A,C,D\}} g(\omega) d\mathbb{P}(\Omega)$?

Solution:

Strategy: This is more for expositional purposes. Note that $g(\omega)$ is a simple function, we can write it as $g(\omega) = 8\mathbb{I}(\omega \in \{A\}) + 25\mathbb{I}(\omega \in \{B\} + 5\mathbb{I}(\omega \in \{C, D\})$. Since it's a simple function, we know $\int_{E \in 2^\Omega} g(\omega) d\mathbb{P}(\Omega) = \sum_{\omega \in E} g(\omega)\mathbb{P}(\omega)$. This is how we'll calculate the two integrals.

Step 1: Rewrite $\mathbb{E}[g(\omega)]$ as an integral and plug in
The definition of $\mathbb{E}[g(\omega)]$ is $\mathbb{E}[g(\omega)] = \int_\Omega g(\omega) d\mathbb{P}(\Omega)$. Plugging in for $g(\omega)$ and $d\mathbb{P}(\Omega)$ and rewriting the integral as in the setup yields

$$\begin{aligned}
\mathbb{E}[g(\omega)] &= \int_\Omega g(\omega)\mathbb{P}(\omega) \\
&= \sum_{\omega \in \Omega} g(\omega)\mathbb{P}(\omega) \\
&= g(A)\mathbb{P}(A) + g(B)\mathbb{P}(B) + g(C)\mathbb{P}(C) + g(D)\mathbb{P}(D) \\
&= 8 * 0.25 + 25 * 0.1 + 5 * 0.6 + 5 * 0.05 \\
&= 2 + 2.5 + 3 + 0.25 = 7.75
\end{aligned}$$

Step 2: Drop B from sum to get the other integral
Note that we just solved for $\sum_{\omega \in \Omega} g(\omega)\mathbb{P}(\omega)$ and now we want to solve for $\sum_{\omega \in \Omega \setminus \{B\}} g(\omega)\mathbb{P}(\omega)$. So we know $\int_{\{A,C,D\}} g(\omega) d\mathbb{P}(\Omega) = [\sum_{\omega \in \Omega} g(\omega)\mathbb{P}(\omega)] - g(B)\mathbb{P}(B)$. So we get $\int_{\{A,C,D\}} g(\omega) d\mathbb{P}(\Omega) = 7.75 - 25 * 0.1 = 5.25$.

3 Integral Example 2

Setup: Let $\Omega = [-1, 1]$ be associated with the Borel σ algebra on that subset and a probability density $f_\Omega(\omega) = |omega|$. Consider a function $g : \Omega \to \mathbb{R}$ given by $g(\omega) = \omega^3$. What is $\mathbb{E}[g]$?

Solution:
Strategy: Again, we're going to use the definition of expectation: $\mathbb{E}[g] = \int_{-1}^1 g(\omega)|\omega| d\omega$.

Solve:

$$\begin{aligned}
\mathbb{E}[g] &= \int_{-1}^1 g(\omega)|\omega| d\omega \\
&= \int_0^1 g(\omega)\omega d\omega - \int_{-1}^0 -g(\omega)\omega d\omega \\
&= \int_0^1 \omega^4 d\omega - \int_{-1}^0 \omega^4 d\omega \\
&= \frac{1}{5}\omega^5 \Big|_0^1 - \frac{1}{5}\omega^5 \Big|_{-1}^0 \\
&= 0
\end{aligned}$$